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Bound states for non-symmetric evolution Schrödinger potentials

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Abstract

We consider the spectral problem associated with the evolution Schrödinger equation, $(D^2 + k^2)\phi = u\phi$, where u is a matrix-square-valued function, with entries in the Schwartz class defined on the real line. The solution ϕ, called the wavefunction, consists of a function of one real variable, matrix-square-valued with entries in the Schwartz class. This problem has been dealt for symmetric potentials u. We found for the present case that the bound states are localized similarly to the scalar and symmetric cases, but by the zeroes of an analytic matrix-valued function. If we add an extra condition to the potential u, we can determine these states by an analytic scalar function. We do this by generalizing the scalar and symmetric cases but without using the fact that the Wronskian of a pair of wavefunction is constant.

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1. Introduction

Bound states have been localized for symmetric potentials [1]. To do this, the fact that the Wronskian is constant is applied. This is key for the existence of the wavefunctions, their uniqueness and the location of the bound states. In the present case, the uniqueness and existence are obtained by solving the Schrödinger evolution equation in two different ways.

We consider the spectral problem for the evolution Schrödinger equation

$$(D^2 + k^2)\phi = u\phi \tag{1}$$

where u is a matrix-valued function, with entries in the Schwartz class defined on the real line, k being a complex number. The solution ϕ consists of a matrix-valued function with entries in the Schwartz class defined on a real line.

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Recent studies involving non-symmetric potentials for problems on half-axis and complex-valued scalar potentials with a real spectrum of eigenvalues are based, in part, on the localization of bound states of non-symmetric evolution potentials (see e.g. [2]).

2. The wavefunctions

Solving by means of Fourier transform, this evolution equation can be transformed into a Volterra integral system. Taking into account the specific asymptotic behaviours $\psi_\pm \sim e^{\pm ikxI_{n\times n}}$ as $x \to \infty$ and $\phi_\pm \sim e^{\mp ikxI_{n\times n}}$ as $x \to -\infty$, we, respectively, obtain the Volterra integral systems

$$\begin{aligned} \psi_\pm &= e^{\pm ikxI_{n\times n}} - \int_x^\infty \frac{e^{ik(x-y)} + e^{-ik(x-y)}}{2ik} u\psi_\pm \, dy \qquad \operatorname{Im} k \geqslant 0 \\ \phi_\pm &= e^{\mp ikxI_{n\times n}} + \int_{-\infty}^x \frac{e^{ik(x-y)} + e^{-ik(x-y)}}{2ik} u\phi_\pm \, dy \qquad \operatorname{Im} k \leqslant 0 \end{aligned} \tag{2a}$$

allowing a way for proving the existence of solutions $\psi_\pm, \phi_\pm$, which are analytic on the upper and lower half k planes, respectively, with the asymptotic behaviour

$$\begin{aligned} \phi_\pm &\sim e^{\mp ikxI_{n\times n}} \qquad x \to -\infty \\ \psi_\pm &\sim e^{\pm ikxI_{n\times n}} \qquad x \to \infty. \end{aligned}$$

Here $I_{n\times n}$ denotes the $n \times n$ identity matrix. The solutions in (2) are known as wavefunctions of the system (1). For the integral systems in (2a), we introduce one reduced wavefunction $m_\pm$ each, which is respectively defined by the relations

$$\psi_\pm = e^{\pm ikx} m_\pm \qquad \phi_\pm = e^{\mp ikx} \hat{m}_\pm.$$

We also obtain the Volterra integral systems for $m_\pm, \hat{m}_\pm$,

$$\begin{aligned} m_\pm &= I_{n\times n} \mp \int_x^\infty \frac{1 - e^{\mp 2ik(x-y)}}{2ik} u m_\pm \, dy \\ \hat{m}_\pm &= I_{n\times n} \pm \int_{-\infty}^x \frac{1 - e^{\pm 2ik(x-y)}}{2ik} u \hat{m}_\pm \, dy. \end{aligned} \tag{2b}$$

These integral systems may be solved by successive approximations, which not only grant the existence and uniqueness of the reduced wavefunctions, but also the analyticity on the half k planes and continuity on their closures. It is clear that all these properties are true for the wavefunctions themselves.

3. Existence and uniqueness of the representation of ϕ_+ in terms of $\psi_\pm$

We also observe that the system in (1) is equivalent to the integral system

$$\Phi = \Phi(x_0) + \int_{x_0}^x \begin{pmatrix} I_{n\times n} & 0 \\ 0 & u - k^2 I_{n\times n} \end{pmatrix} \Phi$$

where

$$\Phi(x_0) = \begin{pmatrix} \phi(x_0) \\ \phi'(x_0) \end{pmatrix}.$$

Such a system may be uniquely solved by successive approximations. So if two solutions and their derivatives agree in a point, they do so on the whole line.

Because of the asymptotic behaviour of the wavefunctions, we find that for large enough x_0 the determinant

$$\det\begin{pmatrix} \psi_-(x_0,k) & \psi_+(x_0,k) \\ \psi_-'(x_0,k) & \psi_+'(x_0,k) \end{pmatrix}$$

is different from zero. Therefore, the linear system in the unknown $a_{ij}(k)$, $b_{ij}(k)$

$$\begin{aligned} \phi(x_0,k) &= \psi_-(x_0,k)a(x_0,k) + \psi_+(x_0,k)b(x_0,k) \\ \phi'(x_0,k) &= \psi_-'(x_0,k)a(x_0,k) + \psi_+'(x_0,k)b(x_0,k) \end{aligned} \tag{3}$$

has a unique solution.

So, $\phi_+(\cdot,k)$ and $\psi_-(\cdot,k)a(x_0,k) + \psi_+(\cdot,k)b(x_0,k)$ are two solutions that agree at x_0 together with their derivatives. Therefore, they agree on the whole real line.

Because of the asymptotic behaviour of $\psi_\pm$, if a linear combination

$$\psi_-c_1 + \psi_+c_2$$

with c_1 and c_2 square matrices, is a singular constant matrix, then $c_1 = c_2 = 0$. Taking this into account, we may conclude that $a(x_0,k) = a(x_1,k)$ for any pair of large real numbers x_0, x_1.

Therefore, we arrive at the following result.

Proposition 1. *Let u and k be a potential and a real number, respectively. Then there exist unique constant matrices $a(k)$ and $b(k)$, such that*

$$\phi_+ = \psi_-a(k) + \psi_+b(k).$$

Because of the asymptotic behaviour, for large enough x, we can multiply by ψ_+^{-1} and $(\psi_+')^{-1}$ the first and second equations in (3), respectively. Subtracting the second equation from the first, we obtain for large enough x, after reducing terms, the following expression:

$$W(\psi_+(\cdot,k),\phi_+(\cdot,k))(x) = W(\psi_+(\cdot,k),\psi_-(\cdot,k))a(k) \tag{4}$$

where the Wronskian matrix of the two derivable matrix-valued functions ϕ and ψ is defined by

$$W(\psi,\phi) = \psi\psi_+'\psi^{-1}\phi - \psi\phi'.$$

In the scalar and symmetric cases, the fact that $a(k)$ can be analytically extended to the upper half k plane follows at once from the fact that the Wronskian is constant.

4. Analytical extension of *a*(*k*)

The asymptotic behaviour of the wavefunctions implies, for all real k and for large enough x, that

$$2\mathrm{i}ka(k) = \lim_{x\to\infty} W(\psi_+(\cdot,k),\phi_+(\cdot,k))(x)$$

uniformly on the upper half k plane. This shows that $2\mathrm{i}ka(k)$ is the uniform limit of analytic functions on the upper half k plane and the uniform limit of continuous functions on its closure, namely,

$$W(\psi_+(\cdot,\cdot),\phi_+(\cdot,\cdot))(x)$$

since the wavefunctions involved are analytic. In summary, we have obtained the following result.

Proposition 2. *$a(k)$ can be analytically extended to the upper half k plane.*

So, the determinant det a is an analytic function on the upper half k plane. Consequently, its zeroes are isolated. This implies that the zeros of $a(k)$ itself are isolated.

From the expressions for the reduced wavefunctions in (2b), we get the continuity and boundness of

$$W(\psi_+(\cdot,k),\psi_-(\cdot,k))$$

on the real line. Consequently, let us extend the Schwartz formula [3] for the upper half k plane to the upper half plane. The relation in (4) now is valid for the upper half k plane since it is valid on the real line. However, we conclude that the asymptotic behaviour of the wavefunctions is invertible. Thus, we have the following result.

Lemma. *For large enough x, the zeros of $a(k)$ are exactly those of*

$$W(\psi_+(\cdot,k)(x),\phi_+(\cdot,k))(x).$$

In other words,

$$a(k)=W(\psi_+(\cdot,k),\phi_+(\cdot,k))(x)=0.$$

5. Determining bound states

In our case, it is still true that bound states can be located by the zeros of $a(k)$ and their location is on the upper half plane as it shows the following result without extra conditions on the potential u.

Theorem. *$a(k_0)=0$ if and only if $\phi(x,k_0)=\psi(x,k_0)c_{k_0}$ for some constant matrix c_{k_0}. In addition, k_0 is on the upper half plane.*

Proof. It is clear that if there exists a constant matrix as indicated in the above theorem, then $W(\psi_+(\cdot,k),\phi_+(\cdot,k))(x)=0$. Thus, the lemma implies that $a(k)=0$.

Since $\psi_+(x,k)$ is invertible for large enough x, we can find a matrix-valued function $c(x,k)$ for large enough x such that

$$\phi_+(x,k)=\psi_+(x,k)c(x,k) \qquad \operatorname{Im}k\geqslant 0. \tag{5}$$

Hence

$$W(\psi_+(x,k),\phi_+(x,k))=W(\psi_+(x,k),\psi_+(x,k)c(x,k))=-\psi_+(x,k)^2\frac{\mathrm{d}c}{\mathrm{d}x}(x,k). \tag{6}$$

Suppose that $a(k_0)=0$. Thus, by the lemma and because $\psi_+(x,k)$ is invertible for large enough x, we get

$$\frac{\mathrm{d}c}{\mathrm{d}x}(x,k_0)=0$$

for large enough x. Then $c(x,k_0)=c_{k_0}$ for large enough x. Consequently, from (5) we get

$$\phi_+(x,k_0)=\psi_+(x,k_0)c_{k_0}$$

on an unbounded open interval of the real line. Both sides of the above equation are solutions of the second-order system (1). This is sufficient to ensure that they agree on the whole real line. Because of the asymptotic behaviour of $\phi_+(x,k_0)$ and $\psi_+(x,k_0)$, we may conclude that $\operatorname{Im}k_0>0$, which proves the theorem. □

6. Conclusions

We have proved that $a(k)$ still determines the bound states for non-symmetric potentials as in the scalar and symmetric cases. This may be done by simple calculations and by using the well-known theory of complex variables and ODEs.

In order to have $\det a(k)$ determining the bound states, it is necessary to impose extra conditions on the potential u. For example, let the nth derivative $a^{(n)}(k)$ be invertible for some $n > 0$ whenever k is on the imaginary axis. This is necessary in order to avoid situations where $\det a(k) = 0$ but $a(k) \neq 0$.

It is well known that linear evolution equations are connected with some nonlinear ones, for example, in our case, with the evolution Korteweg–de Vries (KdV) equation [1, 4]

$$u_t + u_{xxx} - 3(u^2)_x = 0.$$

On the other hand, the localization of bound states has played an important role in recent techniques to solve problems involving non-symmetric potentials on the half-axis and complex-valued scalar potentials with a real spectrum of eigenvalues, which are related to problems involving non-symmetric matrix potentials (see e.g. [2]).

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